ELSEVIER
Discrete Mathematics 232 (2001) 1–9
DISCRETE MATHEMATICS
www.elsevier.com/locate/disc

The ultimate categorical matching in a graph

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Received 24 March 1999; accepted 28 February 2000

Abstract

Let $m(G)$ denote the number of vertices covered by a maximum matching in a graph G. We introduce $m^*(G) = \lim_{n\to\infty} m(G^n)^{1/n}$ where the categorical graph product is used. We show that $m^*(K_{a,b}) = 2\sqrt{ab}$. Moreover, if G is bipartite, with parts A and B, has a matching which saturates A and $A \times B \cup B \times A$ has a perfect matching, then $m^*G = 2\sqrt{|A||B|}$. © 2001 Elsevier Science B.V. All rights reserved.

MSC: 05C70; 05C99

Keywords: Matching; Ultimate categorical matching; Bipartite

1. Introduction

Graph products have been used to find the '*essential*' value of a graph parameter (such as independence number or chromatic number) of a graph G by 'multiplying' G by itself n times and examining the growth of the parameter on G^n. For example, the *Shannon capacity* [8,10,11] of a graph G is defined by

$$\Theta(G) = \lim_{n\to\infty} \beta(\boxtimes_{i=1}^{n} G)^{1/n},$$

where $\boxtimes$ is the strong product of graphs and $\beta(H)$ denotes the independence number of graph H.

Another such concept is the *ultimate chromatic number* of a graph G; that is

$$\chi_u(G) = \lim_{n\to\infty} (\chi(\bullet_{i=1}^{n} G))^{1/n}$$

here $\bullet$ denotes the lexicographic product and $\chi(H)$ the chromatic number of H. This was introduced by Hilton et al. [7] and see also [3,9]. The determination of both the

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[1] Partially supported by a grant from NSERC.

PII: S0012-365X(00)00181-3

Shannon Capacity for some graphs and the ultimate chromatic number for all graphs can be solved using linear programming techniques (see [10] for the former and [5] for the latter).

In this paper we are interested in the behaviour of the ultimate matching number of a graph defined in a similar fashion. Before doing so though, there is another approach one can take and that is to consider the ratio of a parameter to the total number of vertices in G^n. For example if $|V(G)| = k$, the *ultimate independence ratio* of G is

$$I(G) = \lim_{n\to\infty} \beta(\Box_{i=1}^{n} G)/k^n,$$

where $\Box$ is the Cartesian product. This was introduced by Hell et al. [11] (see also [4]). The *ultimate categorical independence ratio* of a graph G which is defined as

$$A(G) = \lim_{n\to\infty} \beta(\times_{i=1}^{n} G)/k^n,$$

was introduced Brown et al. [2]. If one tries this ratio approach with matchings one finds that the sequence is not necessarily monotonic but it appears that all the limit values are either 0 or 1.

Let $G = (V(G), E(G))$ be a graph. We will assume that graphs are finite and simple. We write $a \sim b$ if a is adjacent to but not equal to b, and $a \perp b$ if a is neither adjacent nor equal to b. For a graph G, let $|G|$ be the number of vertices of G and if M is a maximum matching then put $m(G) = 2|M|$, that is $m(G)$ is the number of vertices in a maximum matching. We let $G + H$ denote the *disjoint* union of G and H. A perfect matching is a 1-factor. In this paper we also require $\{1,2\}$-factors, that is a covering of the vertices by disjoint edges and cycles.

The *categorical* product $G \times H$ of G and H has the vertex set $V(G) \times V(H)$ and $(a,x) \sim (b,y)$ if both $a \sim b$ and $x \sim y$. As the categorical product is the only product we shall consider henceforth, for the rest of the paper we use $\times_{i=1}^{n} G$ and G^n interchangeably. Clearly, the categorical product is associative and commutative and distributes over disjoint unions. (For other undefined terms please see [2].)

We define the *ultimate categorical matching* $m^*(G)$ to be

$$m^*(G) = \lim_{n\to\infty} m(G^n)^{1/n}.$$

Our first main result, Proposition 3, is that:
for each graph G, $m^*(G)$ exists and $m^*(G) \leqslant |G|$.

It is easy to see that equality exists when G has a perfect matching. In Theorem 6, we show that equality is also achieved when G has a $\{1,2\}$-factor, i.e. when $V(G)$ can be covered by a set of disjoint edges and cycles. We mainly investigate bipartite graphs. Proposition 7 states

$$m^*(K_{a,b}) = 2\sqrt{ab}.$$

A matching $M = \{(a_i, b_i) \mid i = 1, 2, \ldots, k\}$ *saturates* $A \subset V(G)$ if $|A \cap \{a_i, b_i\}| = 1$ for each $i = 1, 2, \ldots, k$. In Theorems 8 and 9 we show that:

If a *bipartite graph G has a matching which saturates one of its parts A and $A \times (G - A) \cup (G - A) \times A$ has a perfect matching then $m^*(G) = 2\sqrt{|A||G - A|}$.*

This result helps to exactly determine the ultimate categorical matching number for complete multi-partite graphs:

Let G be a complete m-partite graph with parts $A_1, A_2, \ldots, A_m$, where $|A_1| \geqslant |A_2| \geqslant \cdots \geqslant |A_m|$.

1. *If $|A_1| \leqslant |G|/2$ then $m^*(G) = |G|$.*
2. *If $|A_1| > |G|/2$ then $m^*(G) = m^*(K_{|A_1|, \sum_{i>1} |A_i|})$*

2. Proofs

We begin with a pair of results on limits which are required for some of the decomposition theorems. These are easy, and presumably well-known, but we have been unable to find precise statements of them elsewhere. For convenience, all subscripts are assumed to run over the positive integers, and all subscripted variables are assumed to be non-negative reals unless specified otherwise. We use $a_n \to L$ as a short form for the assertion that $\lim_{n\to\infty} a_n$ exists and is equal to L.

Lemma 1. *Suppose that*

$$s_n = \sum_{k=0}^{n} t_{n,k} \quad \text{and} \quad m_n = \max_{k=0}^{n} t_{n,k}.$$

Then

$$s_n^{1/n} \to s \quad \text{if and only if } m_n^{1/n} \to s.$$

Proof. Note that

$$s_n/n \leqslant m_n \leqslant s_n \leqslant n m_n$$

take nth roots, and apply the 'squeeze' theorem. □

We remark (and will later use) that a considerably more general result is available. Namely the number of terms in the sum could be any function of n whose nth root tends to 1.

Corollary 2. *If $a_n^{1/n} \to a$ and $b_n^{1/n} \to b$ and*

$$c_n = \sum_{k=0}^{n} \binom{n}{k} a_k b_{n-k},$$

then $c_n^{1/n} \to a + b$.

Proof. Let

$$t_{n,k} = \binom{n}{k} a_k b_{n-k} \quad \text{and} \quad m_n = \max_{k=0}^{n} t_{n,k}.$$

For any k which is $\mathrm{o}(n)$ (for definiteness, say $k \leqslant \sqrt{n}$),

$$t_{n,k}^{1/n} = b + \mathrm{o}(1) \quad \text{and} \quad t_{n,n-k}^{1/n} = a + \mathrm{o}(1),$$

where the $\mathrm{o}(1)$ term can, and will, be taken to depend on n alone. On the other hand for k between $\sqrt{n}$ and $n - \sqrt{n}$

$$t_{n,k}^{1/n} = \left(\binom{n}{k} a^k b^{n-k} \right)^{1/n} (1 + \mathrm{o}(1)),$$

where again, the $\mathrm{o}(1)$ term may be taken independent of k. From Lemma 1 applied to the binomial theorem we already known that

$$\max_{k=0}^{n} \left(\binom{n}{k} a^k b^{n-k} \right)^{1/n} \to a + b$$

and the boundary values, together with the argument above imply the same for the maximum of the $t_{n,k}^{1/n}$, so again by Lemma 1 we get

$$c_n^{1/n} \to a + b$$

as required. $\square$

Proposition 3. *The sequence $m(G^n)^{1/n}$ converges to a limit $m^*(G) \leqslant |G|$.*

Proof. The result is trivial if G has no edges, so we assume that it does. In particular, $m(G) > 1$. Now observe that if $X = \{(a_i, b_i) \mid i = 1, 2, \ldots, k\}$ is a matching in a graph A and $Y = \{(c_j, d_j) \mid j = 1, 2, \ldots, l\}$ is a matching in a graph B, then $X \times Y$ gives the matching $\{((a_i, c_j), (b_i, d_j)), ((a_i, d_j), (b_i, c_j)) \mid i = 1, 2, \ldots, k,\ j = 1, 2, \ldots, l\}$ in $A \times B$. In particular,

$$m(A \times B) \geqslant m(A) m(B).$$

(This result will be used implicitly or explicitly on many occasions.)

So if we define $a_n = m(G^n)$ we get that

$$1 \leqslant a_k a_l \leqslant a_{k+l} \leqslant |G|^{k+l},$$

for all k and l. Thus $a_n^{1/n}$ is bounded above by $|G|$. We claim that this sequence converges to its supremum $m^*(G)$. For this to fail there must exist some $c < m^*(G)$ and infinitely many n with $a_n^{1/n} \leqslant c$. Choose d and a positive integer t with

$$c < d^{t/(t+1)} < d < m^*(G).$$

Now choose q with $a_q^{1/q} > d$ and $n > tq$ with $a_n < c$. Choose s with $sq \leqslant n < (s+1)q$. Then

$$a_n^{1/n} \geqslant a_{sq}^{1/n} \geqslant a_{sq}^{1/(s+1)q} \geqslant (a_q^{1/q})^{s/(s+1)} \geqslant d^{s/(s+1)} > c$$

a contradiction. $\square$

Theorem 4. *Let* $G = \sum_{i=1}^{n} G_i$. *Then,*

$$m^*(G) \geqslant \sum_{i=1}^{n} m^*(G_i).$$

Proof. Clearly, it is sufficient to consider the case where $n=2$ and we write $G=G_1+G_2$ in this case. Call each vertex of G either a 1-vertex or a 2-vertex in the obvious way. The type of a vertex of G^n is then a sequence of 1's and 2's, representing the type of each of its coordinates. The edges of G^n can only occur between vertices of the same type. After permuting coordinates occasionally

$$G^n = \sum_{k=0}^{n} \binom{n}{k} G_1^k G_2^{n-k}$$

From the proof of Proposition 3, we can construct a matching in $G_1^k G_2^{n-k}$ of size at least $m(G_1^k)m(G_2^{n-k})$. Therefore,

$$m(G^n) \geqslant \sum_{k=0}^{n} \binom{n}{k} m(G_1^k)m(G_2^{n-k}).$$

Now by Corollary 2

$$m^*(G) \geqslant m^*(G_1) + m^*(G_2)$$

as claimed. □

The following proposition may be well-known:

Proposition 5. *The categorical product of two odd cycles is Hamiltonian.*

Proof. Let the cycle lengths be a and b and label the cycles modulo their respective lengths, with the neighbours of vertex x being $x \pm 1$. Suppose $a \leqslant b$. Fix the sequence

$$1, -1, 1, -1, 1, \ldots, -1, 1$$

of length b. Now beginning from vertex $(0,0)$ proceed along a walk by incrementing the first coordinate from the sequence above, repeating it as necessary, and always incrementing the second coordinate by 1. Clearly this returns to $(0,0)$ after ab steps. It does not return there earlier, since any return point must be a multiple of b (from the second coordinate), and vertex 0 is not visited in the first coordinate except in the first and ath pass through the sequence of ± 1's. □

Theorem 6. *Let* G *be a graph with a* $\{1,2,\}$-*factor. Then* $m^*(G) = |G|$.

Proof. Certainly $m^*(G)$ is bounded below by $m^*(H)$, where H is the $\{1,2\}$-factor of G. However H is the sum of a graph with a perfect matching, and some odd cycles. From the proposition above, any power of an odd cycle in Hamiltonian, hence has a matching omitting only one vertex. Thus $m^*(C) = |C|$ for any cycle C (even cycles have perfect matchings!). Now by Proposition 4, $m^*(H) = |H| = |G|$ as we require. $\square$

Proposition 7. *Let $K_{a,b}$ be the complete bipartite graph with parts of size a and b, respectively. Then*

$$m^*(K_{a,b}) = 2\sqrt{ab}.$$

Proof. Suppose, for convenience, that $a \leqslant b$. Let A and B be the two parts of $K_{a,b}$ (of the obvious sizes). As above, we can associate a type with each vertex of $K_{a,b}^n$ (a sequence of A's and B's). Its neighbours are all of the opposite type, i.e. interchanging A's and B's – and all such vertices are its neighbours. In other words,

$$2K_{a,b}^n = \sum_{k=0}^{n} \binom{n}{k} K_{a^k b^{n-k}, a^{n-k} b^k}.$$

In each term, the maximum matching saturates the smaller part, so has size twice that of the smaller part (which is the part with the larger exponent on a.) By Lemma 1, we need only worry about the maximum such term. This occurs when k is as close to $n/2$ as possible. Assuming without loss of generality that n is even, say $n = 2m$, the matching size in this term is

$$\binom{n}{m} 2a^m b^m.$$

The desired result now follows since, as is well known, or from Lemma 1 with $a=b=1$ applied to the binomial theorem, $\binom{n}{m}^{1/n} \to 2$. $\square$

Theorem 8. *Let G be a bipartite graph, with parts A and B, which satisfies the following conditions*:

1. *G has a matching which saturates A.*
2. *In G^2 the subgraph consisting of those vertices with one coordinate from A and the other from B has a perfect matching.*

Then,

$$m(G^n) = m(K_{a,b}^n)$$

and, *in particular*,

$$m^*(G) = 2\sqrt{|A||B|}.$$

Proof. It suffices to check that among the vertices of G^n of a specific type (and its opposite) there is a matching which saturates the smaller part. Thus, we are considering the vertices of the form $A^k B^{n-k}$ and $B^k A^{n-k}$ (and isomorphic copies of this obtained by permuting coordinates) for each $k \leqslant n/2$. In this set, on the first $2k$ coordinates we have a copy of $A^k B^k$ and $B^k A^k$ and here we have a perfect matching using the product of k copies of the matching given in the second condition above. On the remaining coordinates we have a copy of A^{n-2k} and B^{n-2k} and here we have a matching saturating the smaller part. Taking the product of these two matchings we get a matching of the type of vertices in question which saturates the smaller part. □

We remark that the second condition is unduly restrictive, and we could obtain similar results (at least the asymptotic consequences) from a perfect matching of any A^kB^k to B^kA^k pair. However, we do not even know of an example where the cube has a perfect matching but the square does not.

Theorem 9. *Let G be as in Theorem 8. Let H be obtained from G by adding some edges between vertices in A. Then for all n, $m(H^n) = m(G^n)$.*

Proof. Let X be a maximum matching of G^n and let Y be a maximum matching of H^n sharing as many edges as possible with X. For the sake of contradiction, suppose that there is a vertex v_0 matched in Y (to v_1) but not in X. The type of v_0 necessarily includes more B's than A's (since all other vertices are matched in X). Since the neighbours of a B are all A's in either G or H, v_1 necessarily includes more A's than B's, hence is matched in X to some vertex v_2 (which, since this matching is along an edge in G^n, definitely includes more B's than A's again). Continue this process for as long as possible taking v_{2i+1} to be the Y-match (if such exists) of v_{2i}, and v_{2i+2} to be the X-match of v_{2i+1}. The process terminates with some v_{2k} which is not matched in Y. (We can never form a cycle since v_0 is not X-matched, and at the odd step, we can always be certain of finding an X-match.) Now remove all the edges $\{v_{2i}, v_{2i+1}\}$ for $0 \leqslant i < k$ from Y and replace them with $\{v_{2i-1}, v_{2i}\}$ for $1 \leqslant i \leqslant k$. This does not affect the size of Y but does increase the number of edges in common with X thus giving a contradiction. □

Corollary 10. *Let G be a complete m-partite graph with parts $A_1, A_2, \ldots, A_m$ where $|A_1| \geqslant |A_2| \geqslant \cdots \geqslant |A_m|$.*

1. *If $|A_1| \leqslant |G|/2$ then $m^*(G) = |G|$.*
2. *If $|A_1| > |G|/2$ then $m^*(G) = m^*(K_{|A_1|, \sum_{i>1} |A_i|})$.*

Proof. In the first case, G is Hamiltonian since for all vertices x of G, the degree of x is greater than $|G|/2$. Hence $m^*(G) = |G|$. In the second case, G can be obtained from $K_{|A_1|, \sum_{i>1} |A_i|}$ by adding edges in the smaller part, and so Theorem 9 applies. □

3. Example

There are values other than $2\sqrt{|A||B|}$ than can occur as the ultimate categorical matching number for bipartite graphs.

To see this, let G be the graph given in Fig. 1. Let H be the copy of $K_{1,1} + K_{1,2}$ obtained from G by deleting the middle edge. Let X be a maximum matching in H^n

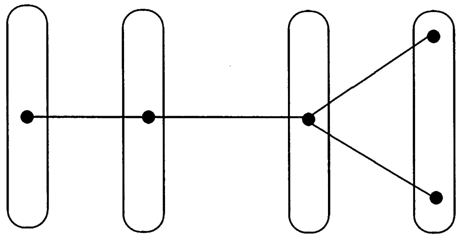

Fig. 1.

and let Y be a maximum matching in G^n that has the maximum number of edges in common with X. Note that an edge in X has vertices of type (after permuting coordinates) $A^aB^bC^cD^{n-a-b-c}$ and $A^bB^aC^{n-a-b-c}D^c$. Note that any vertex with more C's than D's is incident with an edge of X.

If $|Y|>|X|$ then there exists a vertex x_0 on an edge of Y that is not on an edge of X. This necessarily has more D's than C's. Now x_0 is matched, by Y with a vertex x_1 which has more C's than D's. This lies on a matching edge of X the other vertex is x_2 which has more D's than C's. We continue alternating between edges of Y and of X until no new vertices can be found. Then we must have finished with an edge from X. The path P formed has alternate edges from Y and X starting in Y and finishing in X. Form a new matching Y' by taking the edges of Y but switching the edges along the path P for the edges of X. This new matching has the same cardinality as Y but has more edges in common with X. This contradiction shows that $|X|=|Y|$.

Since $K_{a,b}\times K_{c,d}=K_{ac,bd}+K_{ad,bc}$, then it follows that

$$(K_{1,1}+K_{1,2})^n=\sum_{i=1}^{n}\binom{n}{i}K_{1,1}^iK_{1,2}^{n-i}=\sum_{i=1}^{n}\binom{n}{i}2^iK_{1,1}\sum_{j=1}^{n-i}\binom{n-i}{j}K_{2^j,2^{n-j-i}},$$

multiplying through by $K_{1,1}$ gives

$$(K_{1,1}+K_{1,2})^n=\sum_{i=1}^{n}\binom{n}{i}2^{i+1}\sum_{j=1}^{n-i}\binom{n-i}{j}K_{2^j,2^{n-j-i}}.$$

Therefore, the matching number is

$$m((K_{1,1}+K_{1,2})^n)=\sum_{i=1}^{n}\sum_{j=1}^{n-i}\binom{n}{i}2^{i+1}\binom{n-i}{j}m(K_{2^j,2^{n-j-i}}).$$

Since this is a sum of fewer than n^2 terms, we may calculate $m^*(H)$ (and hence $m^*(G)$) by finding the largest term in this sum, and the limiting value of its nth root.

After eliminating a pair of irrelevant factors of 2, it is clear that this largest term is of the form

$$\binom{n}{i}\binom{n}{(n-i)/2}2^i2^{(n-i)/2},$$

for some i. Note that the largest term of the trinomial expansion of

$$(2+\sqrt{2}+\sqrt{2})^n$$

is also of this form. Thus by an argument like that of Corollary 2, it follows that

$$m^*(G)=m^*(H)=2+2\sqrt{2}.$$

4. Questions

Define an edge of a graph G to be *inessential* if it is not included in any maximum matching. Theorem 9 shows adding an edge does not change the ultimate categorical number. In the example of the last section, deleting the inessential edge did not change the final result.

Question 1. *Is it true that deleting inessential edges does not change the ultimate categorical number*?

If G is a graph with no edges then $m^*(G\times H)=m^*(G)m^*(H)=0$ since the product graph has no edges either.

Question 2. *Is there a graph G, with at least one edge, such that for all graphs H, $m^*(G\times H)=m^*(G)m^*(H)$?*

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